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ABSTRACT

This paper examines some of the manipulative aids used in elementary mathematics instruction from the point of view of the cognitive structures need d to understand them. The number line, measurement activities, models for place value, and models for fractions are discussed. (SD)

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ABSTRACT

A careful analysis is given to various manipulative aids that are often used in elementary mathematics instruction. The aids are analyzed in relation to children's perceptual problems related to developmental levels. Emphasis is given to simple aids, where the children "act" on the aids and gradually build internal structures or models that lead to the children's understanding of mathematical concepts.

For the past several years we have been in a movement towards an activity-based curriculum, particularly with heavy emphasis upon manipulative aids, for children. The primary focus of this paper is to examine models that help children build beginning concepts, particularly models that help children move from the preoperational stage to the concrete operational stage of their intellectual development. Such stages vary from concept to concept.

We have been anxious that our children quickly learn many years of patterns and relationships found in the mathematical world. In order to "speed up" the children's learning, well organized models have often been given to the children. Too little attention has been paid to what models are meaningful to children. We feel that if we give them a well organized model, the children will immediately understand and be able to use the model in a meaningful way. Some models, in fact, "cover" mathematics, while others "uncover" mathematics.

Children's developmental levels have been ignored too often in selecting appropriate models for the children. Many of the models given to them may well only be mechanical devices for helping children get answers rather than build understanding.

Flavell stressed:

> It just seems natural to assume that we see space as it is and have always seen it that way. What Piaget wants to stress is (1) that this effortless seeing is really the end product of long and arduous developmental construction, and (2) that construction itself is more dependent upon actions than upon perceptions <u>per se</u>. (p. 328, 1963)

It is the position of this paper that children need to start

with simple models that allow each of the parts to be manipulated in order for the child to fully conceptualize what is happening. If the child is given a well organized model, then he will not understand the parts involved in the model. There are many levels of abstraction in different models; those levels of abstraction are to be examined.

ASSUMPTIONS

The following assumptions are being made for this paper:

(1) Children perceive long before they conceive ideas;

(2) Perceptions do not necessarily lead to conceptions;

(3) Children learn from their own actions, first actions on objects, later actions of thought;

(4) Children need to bring organization to their own world;

(5) In order for the children to bring organization to their world, children need to be given meaningful models that can be manipulated in such a way that the children can build mental images of the ideas to be conceived;

(6) Children need to see the individual parts in a model, and act on the individual parts of the model in order to construct images of the total model.

The above assumptions are supported in literature. (Flavell, 1963; Furth, 1969; Piaget, 1970; Piaget, 1973)

BACKGROUND

As a child goes through school in the elementary grades, he may pass through three stages of development. Those stages are: preoperational, concrete operational, and formal operational. The preoperational stage is a stage during which the child is operating primarily at the perceptual level; he is easily distracted by irrelevant information. He is lacking in the ideas

of invariance, invariance of number, length, area, wholeness, volume, and weight - to name a few. A child may be given five objects and asked how many there are. He counts them and responds that there are five (assuming that he can count meaningful). The same objects are then moved, i.e., changed in position; the child is again asked how many objects are there...The child then recounts all over again, as if it is a completely new problem. He has not been able to hold the idea of fiveness in his head, that is, that five is five no matter how the objects are arranged.

The "standard" test for conservation of number (or invariance) is to show the child five objects, say, arranged as in row 1 in Figure 1 and another five objects arranged as in row 2 in Figure 1. Then the child is asked if there are the same number of objects in each row. A nonconserver of number will not see that there are the same number of objects in each row.

Insert Figure 1 here.

Similarly, a child can be shown two sticks of the same length, and asked to compare their lengths. After lining them up adjacent to each other, the child can verify perceptually that they are the same length. The sticks are then changed in position so that they look different lengths perceptually. The child in a preoperational stage will claim that the sticks are now of different lengths.

The concrete operation stage is when the child's thinking is now operational and relating to concrete objects. He is no longer distracted by irrelevant information. His thinking is related though to the world about him, rather than abstract.

The formal operation stage occurs when the child is able to think more abstractly and to generate hypotheses of his own. This stage supposedly occurs at about the age of 11 or 12. It could occur

earlier for some children; while for others, it may never occur.

Considerably more information concerning these stages of development is available through the listed references.

It is the position of this paper that when a child does not have the invariant ideas he should continue manipulating objects until he sees, through consistency in size, shape, etc., the invariant properties. This has too often been ignored in young children's mathematical development. Many first graders do not have these invariant ideas, and instead of being given the appropriate kinds of manipulative experiences, he is immediately being asked to find answers to such problems as "2 + 3 = ____" in abstraction.

EXAMPLES

A number of examples of various manipulative aids are examined to illustrate the points mentioned above. The examples selected reflect the normal models used in the usual elementary mathematics curriculum. Included are: the number line, models used for measurement, models used for discrete number systems, models used for place value, and models used for fractional numbers.

Example # 1: The Number Line

Many reports (e.g., Cambridge Conferences (1963, 1967, 1969), SMSG (1961)) have recommended that the number line be taught in the first grade. Let's examine this model carefully. Of course, there are many number lines: numbers can be arranged in a random order, numbers can be arranged in the "normal" order but distance is ignored, to name a few types. When reports recommend that "the"

number line be taught in the first grade, however, I am certain that they are referring to a one-to-one correspondence between the real numbers and the points on a line, with distance preserving, in other words, the Euclidean one-space.

The number line is a really nice model in the sense that it leads to a model of the real numbers. It can be used as a model for addition, subtraction, multiplication, and division, to name a few uses. It's uses extend well into high order mathematics and is fundamental in the study of real variables.

But is the number line a part of the young child's world? I think not! The number line involves classification of segments of the same lengths, seriation of segments of different lengths, and conservation of length...to name a few properties. In other words, it is an extremely well organized model.

I have interviewed a number of 6 and 7 year olds, concerning their concepts of a ruler (a more concrete form of the number line), and have observed their uses of the number line. I gave the children rulers such as in Figure 2, and asked them which they would rather use for measurement. I found an interesting relationship between the children who were conservers of length and those who immediately stated that they would want to use Ruler B because the spaces were even.

Insert Figure 2 here.

The nonconservers of length just as likely picked Ruler A because it had more marks, was longer, was shorter, or other various dominate reasons. Some selected Ruler B, but also gave dominate reasons, rather than observing the spacing of the marks.

If the children are not seeing the even spacing involved in the number line, then is it a meaningful model for them?

I have observed a number of children using the number line. They seem to have considerable confusion as to whether they are to focus upon the points (labeled by the whole numbers) or the spaces. The uses that I have observed have been mechanical rather than meaningful.

To add to further confusion, many of the popular textbooks use "hops" on the number line. (See Figure 3.) In chatting with

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Insert Figure 3 here.

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some of my 6 and 7 year old friends, I have found that the hops, themselves, create considerable confusion. I have asked some of these youngsters if you hop this path (see Path A in Figure 4) and you hop this path (see Path B in Figure 4)then have you hopped the same distance? "No" they say!

Insert Figure 4 here.

In addition, I have asked them if you hop from 0 to 3, then from 3 to 5, would that be the same as if you hop from 0 to 5? (See Figure 5.) Again a strong "No!" comes back.

Insert Figure 5 here.

I have discussed the number line with a number of teachers. They admit that they have observed a number of problems with it

also. But the scope and sequence demands that they cover it, so they do. They also have noted the problems with the hops, and several have told me that they make their youngsters show their hops from whole number to consecutive whole number, rather than taking long hops.

The hops have completely missed the point of the use of the number line as a model! The number line is a good model, when meaningful, because it uses the Euclidean property that if point B belongs to segment AC then the measure of segment AB plus the measure of segment BC is equal in length to segment AC. It has nothing to do with hops!! The focus should be upon lengths, not hops!

Am I suggesting that the number line be omitted from the elementary curriculum altogether? No, by no means! But I am suggesting that there are some serious problems with this premature use of the number line.

How would I want to see the number line developed? I would want to see the number line come out of a linear measurement model (more on that in Example # 2). In this measurement model, I would want the child to construct his own ruler (when it is meaningful to him) and then use the same idea to construct his own number line, which thus would represent internalized learning. I would want to see the youngsters use their number lines with the emphasis upon lengths rather than upon hops. To follow the suggestions that I have given, the child probably will not be ready to construct his number line as soon as the curriculum now requests. Among other things, conservation of length is linked into the approach.

Example # 2: Models used for Measurement

Today there is a trend to use measurement as an approach to the elementary mathematics curriculum. The feeling is that measurement is far more concrete and manipulative for the children. I fully agree! But at the same time, there are some problems.

First of all, the children should be made to see the difference between a discrete number system and a continuous number system. Counting is very different from weighing, comparing lengths, etc. If one is to compare two discrete sets, one can create a one-to-one matching between the elements of the sets. If one is to compare weights, lengths, etc. another procedure is needed.

A typical six year old can do qualitative comparison between objects that can be put together and compared perceptually. For instance, lengths of two objects, that can be lined up next to each other, can be determined perceptually through observation. Such a comparison need not involve operations of thought at an abstract level. At the same time, comparing lengths of objects that can not be lined up next to each other involves the idea of a transitive property and more abstract thinking. The thinking patterns needed for this have been studied by Steffe (1971) and Bailey (1974). They have found the ability to solve such problems depends upon the child's ability to conserve lengths.

The distinction between cardinality and quantity is not immediately obvious to children. This past year I conducted a teaching experiment with some first grade youngsters, in which they first performed the action of building some roads with Cuisenaire rods. Then they were to reflect to tell me who had made the longest road. The

youngsters tried to tell me that the one with the most number of rods (irrespective of length) had the longest road. After several debates, I had to select a simple example using two short Cuisenaire rods (e.g., a red rod and a white rod) lined up together next to a longer road (e.g., an orange rod) then asked them how many rods were there in each case, and which made the longer road?

To only give children well organized and well behaved models would have kept them from seeing this important distinction. Similarly, I would give them varying weights for different sizes.

A second major problem in the use of measurement is the assignment of a number to the quantity being measured. To attach a number to a length, weight, area, or volume means that a unit has been selected, and involves unit iteration. The ability to handle unit iteration abstractly is linked to conservation (Piaget, Inhelder, and Szeminska, 1960). Again there are a number of levels of abstraction involved in the process of unit iteration. A number of present day textbooks for kindergarten and first grade have been surveyed concerning introduction to measurement ideas (particularly linear). It was found that in many cases, either the ruler was immediately given to children or they were given a day or two of experiences involving measuring with paper clips, erasers, etc. then were given the ruler. In each case I feel the ruler came too fast and was a part of the adult's organization rather than the child's organization.

I have interviewed a number of first graders and found that all that I interviewed could do unit iteration with a multiple number of objects, e.g,, paper clips. The problem came in though when the children were given only one unit and were asked to do

an iteration using only the one unit. Note that in the first case the children were able to line all of the parts up at the perceptual level. While in the latter case, using only one unit involved an operation of the mind rather than strickly through perception. To remember where to move the object to repeat the iteration involved remembering where the last iteration stopped. This latter case was found to be linked to conservation of length. Nonconservers of length often just arbitrarily moved the object without paying attention to where they moved it.

In addition, I asked the youngsters to make a ruler for me. Again nonconservers of length got their spacing mixed up, while conservers seemed to be able to construct their rulers correctly.

I would suggest the following sequence of activities for linear measurement (this, of course, would be spread over a long period of time):

(1) Children first be given experiences that help them distinguish between length and cardinality;

(2) Children be given experiences measuring objects using a multiple number of objects (e.g., paper clips, erasers, Cuisenaire rods of the same length);

(3) Children be asked to find the measure of objects when given only one object with which to measure (Note: there is a developmental level involved at this point.):

(4) Children construct their own rulers (representing internalized learning) and use their constructed rulers for measuring;

(5) Children can now construct their own number lines and use them in a meaningful way.

I would suggest that similar activities be constructed for other measurement strands.

I might comment that many six year olds can be trained to use

a ruler. But what is happening in many cases, is that they simply are doing a visual and perceptual comparison, and simply reading off the numeral from the ruler. They are not conceiving the unit iteration ideas. Those ideas come through repeated use of manipulative aids as described above, where the children observe through their own actions, the consistency and size of the units.

Example # 3: Models used for Discrete Sets

By all rights this should probably have been the first example, but I have intentionally waited until the first two examples have been discussed before turning to this one. Discrete sets would probably first come into the child's world, along with qualitative comparison of continuous sets.

In this case the children should have lots of experience with discrete objects, manipulating them, classifying them, and counting them. Thus far I have not discussed printed materials. One of my graduate students tested some six year olds concerning their conservation of number ideas in relation to real objects and pictures of objects. Within the pictures, he varied a number of distractors such as: size, spacing, brightness. Of the children he interviewed, he found that conservation with real objects was a prerequisite to conservation with pictures. Some students were successful with real objects but had difficulty with the pictures. In no case was a child able to handle pictures when he could not handle the conservation ideas with real objects.

Another important idea in working with objects, is the "trading in" idea, that is, replacing an object or set of objects with an

equivalent number of objects. For instance, consider money (which, of course, is a type of measurement model, but different from those discussed in Example # 2.) It is important that a child be able to replace a nickel with 5 pennies, and in a number of cases actually do the action. Some of my undergraduate students were teaching a group of first graders a unit on money. One day as I was visiting them and questioning the youngsters about what they knew, I found that the children could rapidly tell me that a nickel had the same value as 5 pennies, and that a dime had the same value as 10 pennies. But then I gave the children a nickel, a dime, and a penny and asked them the value of those coins altogether. The children could not respond! I suggested to my college students that they bring lots of pennies and have the children physically exchange the nickel for five pennies, etc. The problem seemed to take care of itself then. Seems that the children needed the action of exchanging the coins and have all of the parts available to them in order to find the value.

Cuisenaire rods are often referred to as good aids to "teach" addition and subtraction. These rods, however, are based upon a measurement model and involve a "trading in". Thus I would classify them as being a far more abstract model for working with discrete sets.

Example # 4: Models for Place Value

In discussions with teachers, it has rapidly been revealed that one of their biggest problems in the elementary mathematics curriculum is tied up in place value. In a recent workshop that I had with some elementary teachers, I asked them what were their serious problems

at their individual grade levels. The second grade teachers rapidly identified place value to be their biggest problem. The third grade teachers also identified place value to be their number one problem. Fourth grade teachers also had the same problem, as well as fifth grade teachers!

To date, careful research has not been done to establish whether or not there are developmental problems with place value, but I suspect that there are. Two key ideas that I suspect are major problems with place value are: "trading in" and conservation of the whole. Conservation of the whole involves seeing that the whole and all of it's parts (when properly partitioned) gives the same result. Locked in the place value idea is seeing that 1 ten and 10 ones give the same result, and involves both "trading in" and conservation of the whole numbers.

I might also comment that I have observed children doing exercises from their textbooks such as the follows:

34 = ____ tens + ____ ones

\+ 25 = ____ tens + ____ ones

_____ _____

It was my observation that the children could imitate the format and fill in appropriate responses, but that they really did not understand what it was all about.

Typical models for place value are: sticks or straws bunched together, Dienes blocks, Cuisenaire rods, money, and abaci. Many textbooks show a few pages of pictures of the sticks bunched together, then rapidly move to pictures of abaci. My claim is that the change comes too fast. I feel that the children need lots of experience

manipulating stricks or straws...bundling and unbundling, before they are ready to move to deeper levels of abstraction. I have observed some fourth graders very happily manipulating straws to solve subtraction problems with "borrowing" and very happy to do it because for the first time they have been able to solve such problems.

Another nice aid at the low level of abstraction is unifix cubes. Unifix cubes are particularly nice in the sense that they can easily be popped together or taken apart so that the children can easily see the whole and the parts.

Dienes blocks are a little more abstract because they involve a "trading in" of a "long" for 10 "little cubes", or a "flat" for 10 "longs", etc. Cuisenaire rods for this purpose are even more abstract because the units are not physically in front of the child. Money also involves a "trading in" but money is different from some of the other models, because it is closer to the child's real world.

The abacus seems to me to be the most abstract of the aids mentioned. As one second grade teacher told me, she felt that it was more abstract than the symbols themselves. It does not provide for the immediate breaking down into parts such as with the sticks. Also the "trading in" is not obvious.

Example # 5: Models for Fractional Numbers

The fractional number system is far more complex than the whole number system. Furthermore, it has been my observation in working with secondary students and college students that there are many people who probably NEVER understand this system.

Two key developmental problems, among others, have been identify as being linked to fractional numbers. Those two key ideas are: conservation of the whole and conservation of area (Piaget, Inhelder, and Szeminska, 1960). Furthermore, it has been noted that rectangular models for fractional numbers are easier for children than circular models.

One of my college students was teaching a unit on concept of fractions with some fourth graders. They first used Cuisenaire rods then they moved to pattern blocks. The college students noted that the move from Cuisenaire rods to the pattern blocks was very difficult for the children. Finally, they had to back up to the Cuisenaire rods and try to draw the analogy between the two models. Note that the student's finding was consistent with Piaget's et al.'s finding mentioned above.

Looking at levels of abstraction for fraction models, one would first want to do lots of work with measurement, both linear and area, as well as possibly others. The children need to see all the parts first. Then he could move to a "trading in" type model, such as with the Cuisenaire rods or pattern blocks. Finally, he could use the number line. The number line should, however, enter in only after the child has worked for quite a while with more manipulative models.

I might add that too few textbooks relate measurement with their treatment of fractions. To work with the concept of fractions, one first needs to define a unit and then work with unit iteration.

I might also add that the concept is taken too lightly and emphasis is place too rapidly upon operations with fractions (or rational

numbers) which in turn becomes strickly symbol manipulation.

ADDITIONAL COMMENTS

The above discussion has centered around, primarily, the use of manipulative aids. Little has been said about the use of pictures and/or symbols. This has been done primarily because I feel that too little emphasis and understanding has been placed on the role of manipulative aids. Now I would like to make a few comments about pictures and symbols.

Pictures

As mentioned in Example # 3, some investigation has been started looking at children's concepts of objects in relation to pictures. It is suspected that children need to reach conservation in relation to real objects before pictures will be meaningful to them. Furthermore, it is felt that children should be allowed to manipulated pictures of objects rather than just look at pictures in a well organized workbook. For example, allow children to classify pictures of objects so that pictures having the same number of objects will be sorted together.

Symbols

Numbers, length. weight, etc. exist irrespective of man. Symbols on the other hand have been created out of man's need to communicate with his fellow men. The symbols are not the ideas, but rather are about the ideas. First, the idea must first exist and be meaningful to the child, e.g., the child must have the idea of fiveness before

"5" is meaningful to him.

Regretfully, too much emphasis has been placed upon symbols rather than the ideas represented by the symbols. Take, for example, place value. Some of the popular textbooks have treated the entire two-digit, three-digit, etc. arithmetic strickly from a symbolic point of view. No wonder children have difficulty with the ideas... first understanding, then symbols!!

How would I want to introduce the use of symbols? I personally feel that the children should perform the actions, make judgements, then record their results...through the process of communicating their results, symbols will eventually be introduced. A need for symbols should be created before they are given to the children.

In discussions with some reading experts, it was revealed that the usual first grader starts first grade with a speaking vocabulary of at least 10,000 words. He then begins to learn the symbols for those words. Often it is not until, he reaches third grade before he sees symbols for words that are new to him. In mathematics we often teach him a new idea and immediately insist upon the symbol for the idea, when in fact the child may not even be ready for the idea.

IMPLICATIONS FOR TEACHING

I have attempted to identify various levels of abstractions of different manipulative aids. I have taken the position that too little attention has been paid to this, and that we have been in too big of a hurry to rush on to abstraction. A number of children have gotten lost along the way.

I have indicated that children should bring organization to their own world. This does not mean that they should be left to

haphazard experiences, but rather that the experiences should be planned to allow for the children to bring organization to their own world. The teacher might serve more as a catalyst for creating situations and selecting appropriate models for the children to use. Secondly, he is needed to provide counterexamples that compel reflections and reconsiderations of over-hasty solutions. (Piaget, 1973, p. 16)

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0 0 0 0 0 0 row 1

0 0 0 0 0 0 row 2

Figure 1

Figure 2

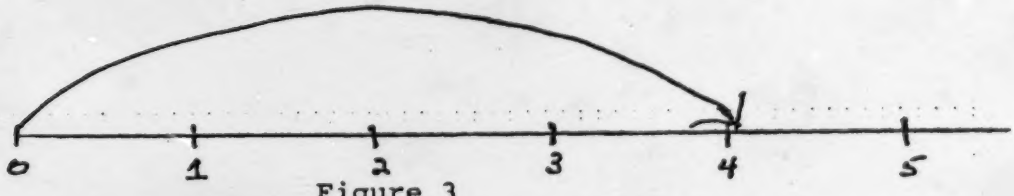

Figure 3

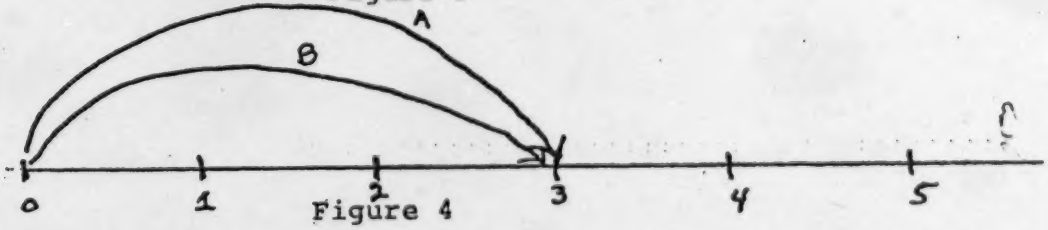

Figure 4

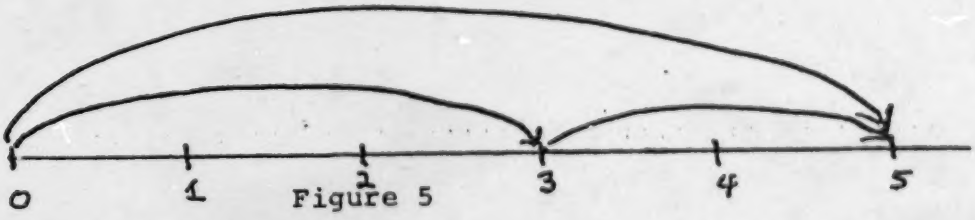

Figure 5